SoftCOM '17
tutorials

Finite-Difference Time-Domain: From basic principles to realistic ground penetrating radar modelling

by

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Finite-Difference Time-Domain: From basic principles to realistic ground penetrating radar modelling

Dr Antonis Giannopoulos

welcome.

Workshop Organisation

- **Section A: GPR forward problem - FDTD**
  - Basic concepts of GPR modelling; Basic concepts of FDTD; Stability; Dispersion; modelling of objects; excitation; modelling errors; pitfalls and problems; advanced topics
- **Section B: gprMax**
  - Introduction to gprMax; History; Development; availability; limitations;

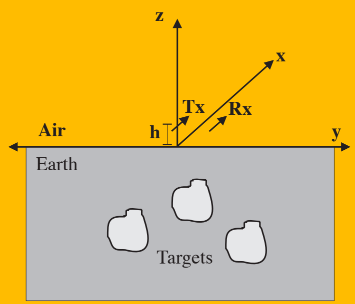
First things first!

GPR modelling

GPR Modelling

What do we need to model ?

(. . .from an electromagnetic point of view . . .)



GPR Modelling

Why we bother ?

Because a GPR model ...

...could improve our understanding and aid the interpretation of complex data sets.

...could be the test-bed for new processing techniques or could be used to assess the effectiveness of existing ones.

...could be used as part of an automatic interpretation process. In other words help us to solve the inverse problem.

GPR Modelling

How can we do it ?

By simply solving Maxwell's equations which are the governing equations for the GPR forward problem. The solution is found subject to the initial and boundary conditions.

Maxwell's equations

Faraday's law

$$\oint_C \mathbf{E} \cdot d\hat{\mathbf{l}} = -\iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\hat{\mathbf{s}}$$

Ampere-Maxwell law

$$\oint_C \mathbf{H} \cdot d\hat{\mathbf{l}} = \iint_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\hat{\mathbf{s}} + \iint_S \mathbf{J_c} \cdot d\hat{\mathbf{s}} + \iint_S \mathbf{J_s} \cdot d\hat{\mathbf{s}}$$

Gauss' electric law

$$\oiint_S \mathbf{D} \cdot d\hat{\mathbf{s}} = \iiint_V q\, dV$$

Gauss' magnetic law

$$\oiint_S \mathbf{B} \cdot d\hat{\mathbf{s}} = 0$$

Continuity of electric charge

Not usually part of Maxwell's equations

$$\oiint_S \mathbf{J} \cdot d\hat{\mathbf{s}} = -\iiint_V \frac{\partial q}{\partial t} dV$$

Maxwell's Integral Equations

$$\oint_C \mathbf{E} \cdot d\hat{\mathbf{l}} = -\iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\hat{\mathbf{s}}$$

$$\oint_C \mathbf{H} \cdot d\hat{\mathbf{l}} = \iint_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\hat{\mathbf{s}} + \iint_S \mathbf{J_c} \cdot d\hat{\mathbf{s}} + \iint_S \mathbf{J_s} \cdot d\hat{\mathbf{s}}$$

$$\oiint_S \mathbf{D} \cdot d\hat{\mathbf{s}} = \iiint_V q\, dV \qquad \oiint_S \mathbf{B} \cdot d\hat{\mathbf{s}} = 0$$

With Stoke's and divergence theorems

$$\oint_S \mathbf{A} \cdot d\hat{\mathbf{l}} = \iint_S (\nabla \times \mathbf{A}) \cdot d\hat{\mathbf{s}}$$

$$\oiint_S \mathbf{A} \cdot d\hat{\mathbf{s}} = \iiint_V (\nabla \cdot \mathbf{A}) dV$$

we get the differential forms ....

Faraday's law

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

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Ampere-Maxwell law

$$\boldsymbol{\nabla} \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J_c} + \mathbf{J_s}$$

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Gauss' electric law

$$\boldsymbol{\nabla} \cdot \mathbf{D} = q_v$$

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Gauss' magnetic law

$$\nabla \cdot \mathbf{B} = 0$$

Maxwell's Differential Equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J_c} + \mathbf{J_s}$$

$$\nabla \cdot \mathbf{D} = q_v \qquad \nabla \cdot \mathbf{B} = 0$$

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Constitutive Relations for fields

$$\begin{aligned} \mathbf{D} &= \bar{\bar{\epsilon}} * \mathbf{E} \\ \mathbf{B} &= \bar{\bar{\mu}} * \mathbf{H} \\ \mathbf{J_c} &= \bar{\bar{\sigma}} * \mathbf{E} \end{aligned}$$

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For simple cases where the electrical properties can be assumed to be frequency independent the convolutions reduce to multiplications. For isotropic media the tensors reduce to scalars and computations are simplified.

The fact that the electrical properties can be assumed that do not vary with frequency does not mean that the velocity of propagation or the attenuation of the electromagnetic pulses is frequency independent!

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$$\begin{aligned} \mathbf{D} &= \epsilon \mathbf{E} \\ \mathbf{B} &= \mu \mathbf{H} \\ \mathbf{J_c} &= \sigma \mathbf{E} \end{aligned}$$

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} \qquad \mu_r = \frac{\mu}{\mu_0}$$

How we solve these equations ?

Analytically with great difficult and only for simple cases and geometries. Lots of Brain Power little Computing Power!

Numerically with relative ease for more realistic problems and complex geometries. Lots of Computing Power not much Brain Power!

A. Sommerfeld, M.J. Strutt, H. Weyl, R.K. Moore, C.T. Tai, H. Ott, J.R. Wait, R.W. King, A. Baños, A.P. Annan and many more ...

Which numerical method?

Integral Formulations

$$\int$$

Differential Formulations

$$\partial$$

Which numerical method?

Frequency Domain **Time Domain**

$$j\omega \qquad \partial/\partial t$$

Which numerical method?

Implicit **Explicit**

$$G(Y(t), Y(t+\Delta t)) = 0 \qquad Y(t+\Delta t) = F(Y(t))$$

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FDTD

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M. Moghaddam,W.C. Chew,T. Wang, G. W. Hohmann, Roberts R., Daniels J.J., Bourgeois, G. Smith, L. Gurel, T. Bergmann, K. Holliger, F. L. Teixera, N. Cassidy and many more ...

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FDTD

the

good

FDTD

the

bad

FDTD

and

the

ugly

How it works

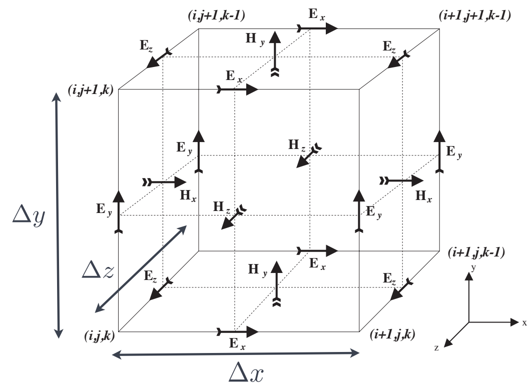
Space is discretised using small cuboid cells called **Yee Cells**.

The electromagnetic field components are arranged in a regular fashion depending on the geometry of the Yee cell.

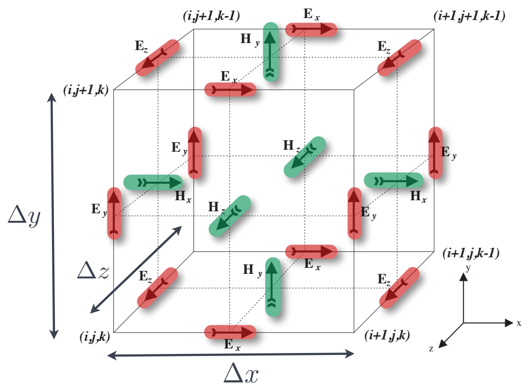
There are no field components co-located in space. They are all arranged in a staggered fashion.

Material properties are assigned at the location of relevant electromagnetic field components.

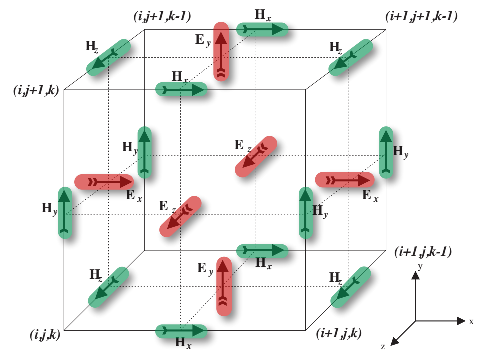
The Yee cell



Space



Space



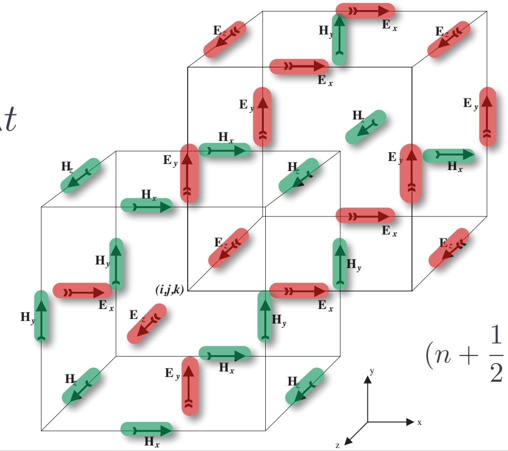
Time is discretised and the solution advances with small steps until it reaches the required time window.

The electromagnetic field components are staggered in time as well. The electric fields are computed half a time-step apart from the magnetic fields.

This type of numerical integration of the underlying governing equations is called a leapfrog system.

Time

$n\Delta t$



$(n+\frac{1}{2})\Delta t$

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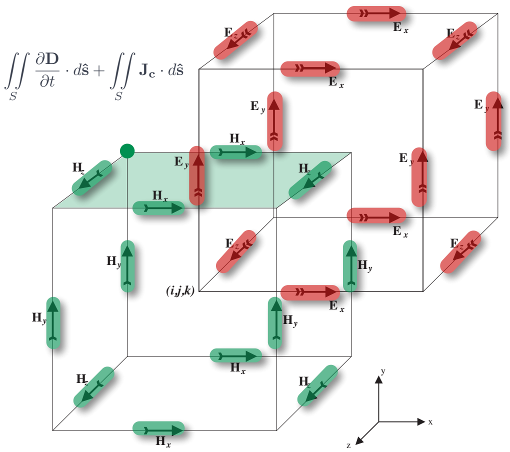
$$\oint_C \mathbf{H} \cdot d\hat{\mathbf{l}} = \iint_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\hat{\mathbf{s}} + \iint_S \mathbf{J_c} \cdot d\hat{\mathbf{s}}$$



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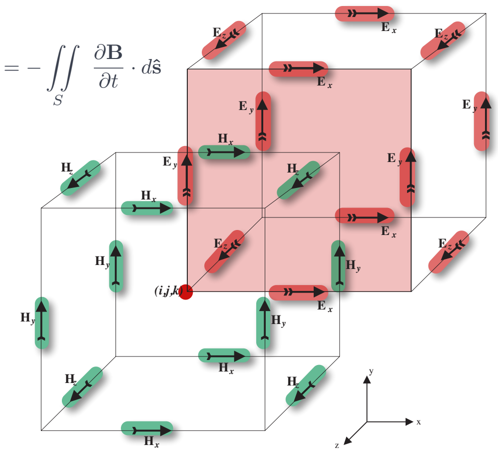
$$\oint_C \mathbf{E} \cdot d\hat{\mathbf{l}} = -\iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\hat{\mathbf{s}}$$



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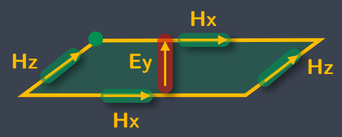
$$\iint_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\hat{\mathbf{s}} + \iint_S \mathbf{J_c} \cdot d\hat{\mathbf{s}} = \oint_C \mathbf{H} \cdot d\hat{\mathbf{l}}$$



$$\epsilon(i,j+1/2,k)\frac{E_y\big|^{n}_{(i,j+1/2,k)} - E_y\big|^{n-1}_{(i,j+1/2,k)}}{\Delta t}\Delta x \Delta y + \sigma(i,j+1/2,k) E_y\big|^{n-1/2}_{(i,j+1/2,k)} \Delta x \Delta y =$$
$$H_x\big|^{n-1/2}_{(i,j+1/2,k+1/2)} \Delta x - H_x\big|^{n-1/2}_{(i,j+1/2,k-1/2)} \Delta x - H_z\big|^{n-1/2}_{(i+1/2,j+1/2,k)} \Delta z + H_z\big|^{n-1/2}_{(i-1/2,j+1/2,k)} \Delta z$$

$$\epsilon(i,j+1/2,k)\frac{E_y\big|^{n}_{(i,j+1/2,k)} - E_y\big|^{n-1}_{(i,j+1/2,k)}}{\Delta t} + \sigma(i,j+1/2,k) E_y\big|^{n-1/2}_{(i,j+1/2,k)} =$$
$$\frac{H_x\big|^{n-1/2}_{(i,j+1/2,k+1/2)} - H_x\big|^{n-1/2}_{(i,j+1/2,k-1/2)}}{\Delta z} - \frac{H_z\big|^{n-1/2}_{(i+1/2,j+1/2,k)} - H_z\big|^{n-1/2}_{(i-1/2,j+1/2,k)}}{\Delta x}$$

$$\sigma(i,j+1/2,k) E_y\big|^{n-1/2}_{(i,j+1/2,k)} = \sigma(i,j+1/2,k)\frac{E_y\big|^{n}_{(i,j+1/2,k)} + E_y\big|^{n-1}_{(i,j+1/2,k)}}{2}$$

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$$E_y\big|^{n}_{(i,j+1/2,k)} = \left(\frac{1 - \frac{\sigma(i,j+1/2,k)\Delta t}{2\epsilon(i,j+1/2,k)}}{1 + \frac{\sigma(i,j+1/2,k)\Delta t}{2\epsilon(i,j+1/2,k)}}\right) E_y\big|^{n-1}_{(i,j+1/2,k)} +$$
$$\left(\frac{\frac{\Delta t}{\epsilon(i,j+1/2,k)}}{1 + \frac{\sigma(i,j+1/2,k)\Delta t}{2\epsilon(i,j+1/2,k)}}\right)\left(\frac{H_x\big|^{n-1/2}_{(i,j+1/2,k+1/2)} - H_x\big|^{n-1/2}_{(i,j+1/2,k-1/2)}}{\Delta z} - \frac{H_z\big|^{n-1/2}_{(i+1/2,j+1/2,k)} - H_z\big|^{n-1/2}_{(i-1/2,j+1/2,k)}}{\Delta x}\right)$$

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Differential forms of Maxwell's equations

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$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J_c} + \mathbf{J_s}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\begin{aligned}
\epsilon \frac{\partial E_x}{\partial t} + \sigma E_x &= \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\
\epsilon \frac{\partial E_y}{\partial t} + \sigma E_y &= \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\
\epsilon \frac{\partial E_z}{\partial t} + \sigma E_z &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \\
\mu_0 \frac{\partial H_x}{\partial t} &= \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\
\mu_0 \frac{\partial H_y}{\partial t} &= \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\
\mu_0 \frac{\partial H_z}{\partial t} &= \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}
\end{aligned}$$

$$\epsilon \frac{\partial E_y}{\partial t} + \sigma E_y = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}$$

Finite difference approximations

$$\frac{d\Psi}{dh} = \lim_{\Delta h \to 0} \frac{\Psi(h_1) - \Psi(h_2)}{\Delta h} \qquad \Delta h = h_1 - h_2$$

$$\Psi(x + \Delta x, y, z, t) = \Psi(x, y, z, t) + \Delta x \frac{\partial \Psi(x, y, z, t)}{\partial x} + \frac{(\Delta x)^2}{2!} \frac{\partial^2 \Psi(x, y, z, t)}{\partial x^2} + \frac{(\Delta x)^3}{3!} \frac{\partial^3 \Psi(x, y, z, t)}{\partial x^3} + O[(\Delta x)^4]$$

$$\Psi(x - \Delta x, y, z, t) = \Psi(x, y, z, t) - \Delta x \frac{\partial \Psi(x, y, z, t)}{\partial x} + \frac{(\Delta x)^2}{2!} \frac{\partial^2 \Psi(x, y, z, t)}{\partial x^2} - \frac{(\Delta x)^3}{3!} \frac{\partial^3 \Psi(x, y, z, t)}{\partial x^3} + O[(\Delta x)^4]$$

$$\Psi(x + \Delta x, y, z, t) - \Psi(x - \Delta x, y, z, t) = 2\Delta x \frac{\partial \Psi(x, y, z, t)}{\partial x} + \frac{(\Delta x)^3}{3} \frac{\partial^3 \Psi(x, y, z, t)}{\partial x^3} + O[(\Delta x)^5]$$

$$\frac{\partial \Psi(x,y,z,t)}{\partial x} = \frac{\Psi(x+\Delta x,y,z,t) - \Psi(x-\Delta x,y,z,t)}{2\Delta x} + O[(\Delta x)^2]$$

Second Order accuracy of the approximation of the derivative

$$\frac{\partial \Psi(x,y,z,t)}{\partial x} = \frac{\Psi(x+\Delta x/2,y,z,t) - \Psi(x-\Delta x/2,y,z,t)}{\Delta x} + O[(\Delta x)^2]$$

$$\frac{\partial \Psi(x,y,z,t)}{\partial t} = \frac{\Psi(x,y,z,t+\Delta t/2) - \Psi(x,y,z,t-\Delta t/2)}{\Delta t} + O[(\Delta t)^2]$$

$$\epsilon \frac{\partial E_y}{\partial t} + \sigma E_y = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}$$

$$\epsilon(i,j+1/2,k)\frac{E_y|^{n}_{(i,j+1/2,k)} - E_y|^{n-1}_{(i,j+1/2,k)}}{\Delta t} + \sigma(i,j+1/2,k)E_y|^{n-1/2}_{(i,j+1/2,k)} = \frac{H_x|^{n-1/2}_{(i,j+1/2,k+1/2)} - H_x|^{n-1/2}_{(i,j+1/2,k-1/2)}}{\Delta z} - \frac{H_z|^{n-1/2}_{(i+1/2,j+1/2,k)} - H_z|^{n-1/2}_{(i-1/2,j+1/2,k)}}{\Delta x}$$

$$\sigma(i,j+1/2,k)E_y|^{n-1/2}_{(i,j+1/2,k)} = \sigma(i,j+1/2,k)\frac{E_y|^{n}_{(i,j+1/2,k)} + E_y|^{n-1}_{(i,j+1/2,k)}}{2}$$

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$$E_y|^{n}_{(i,j+1/2,k)} = \left(\frac{1 - \frac{\sigma(i,j+1/2,k)\Delta t}{2\epsilon(i,j+1/2,k)}}{1 + \frac{\sigma(i,j+1/2,k)\Delta t}{2\epsilon(i,j+1/2,k)}}\right) E_y|^{n-1}_{(i,j+1/2,k)} + \left(\frac{\frac{\Delta t}{\epsilon(i,j+1/2,k)}}{1 + \frac{\sigma(i,j+1/2,k)\Delta t}{2\epsilon(i,j+1/2,k)}}\right)\left(\frac{H_x|^{n-1/2}_{(i,j+1/2,k+1/2)} - H_x|^{n-1/2}_{(i,j+1/2,k-1/2)}}{\Delta z} - \frac{H_z|^{n-1/2}_{(i+1/2,j+1/2,k)} - H_z|^{n-1/2}_{(i-1/2,j+1/2,k)}}{\Delta x}\right)$$

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$$H_x|^{n+1/2}_{i,j+1/2,k+1/2} = H_x|^{n-1/2}_{i,j+1/2,k+1/2} + \frac{\Delta t}{\mu_0}\left(\frac{E_y|^{n}_{i,j+1/2,k+1} - E_y|^{n}_{i,j+1/2,k}}{\Delta z} - \frac{E_z|^{n}_{i,j+1,k+1/2} - E_z|^{n}_{i,j,k+1/2}}{\Delta y}\right)$$

$$H_y|^{n+1/2}_{i+1/2,j,k+1/2} = H_y|^{n-1/2}_{i+1/2,j,k+1/2} + \frac{\Delta t}{\mu_0}\left(\frac{E_z|^{n}_{i+1,j,k+1/2} - E_z|^{n}_{i,j,k+1/2}}{\Delta x} - \frac{E_x|^{n}_{i+1/2,j,k+1} - E_x|^{n}_{i+1/2,j,k}}{\Delta z}\right)$$

$$H_z|^{n+1/2}_{i+1/2,j+1/2,k} = H_x|^{n-1/2}_{i+1/2,j+1/2,k} + \frac{\Delta t}{\mu_0}\left(\frac{E_x|^{n}_{i+1/2,j+1,k} - E_x|^{n}_{i+1/2,j,k}}{\Delta y} - \frac{E_y|^{n}_{i+1,j+1/2,k} - E_y|^{n}_{i,j+1/2,k}}{\Delta x}\right)$$

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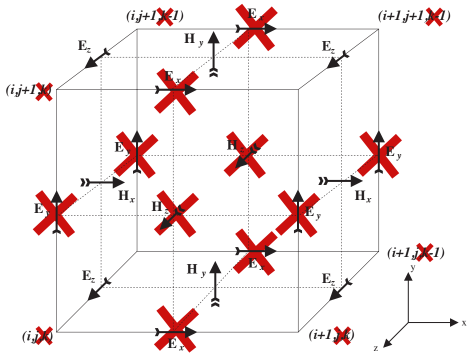
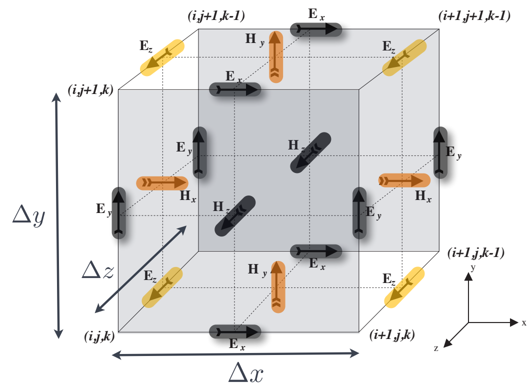
2D

$$
\begin{aligned}
\epsilon \frac{\partial E_x}{\partial t} + \sigma E_x &= \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\
\epsilon \frac{\partial E_y}{\partial t} + \sigma E_y &= \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\
\epsilon \frac{\partial E_z}{\partial t} + \sigma E_z &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \\
\mu_0 \frac{\partial H_x}{\partial t} &= \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\
\mu_0 \frac{\partial H_y}{\partial t} &= \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\
\mu_0 \frac{\partial H_z}{\partial t} &= \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}
\end{aligned}
$$

Transverse Magnetic (TMz)



Transverse Magnetic (TMz)



$$\epsilon \frac{\partial E_z}{\partial t} + \sigma E_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}$$
$$\mu_0 \frac{\partial H_x}{\partial t} = \frac{\partial E_z}{\partial y}$$
$$\mu_0 \frac{\partial H_y}{\partial t} = -\frac{\partial E_z}{\partial x}$$

Stability

... there is no free lunch ...

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Courant Stability Condition

FDTD is a march on time electromagnetic solver. It propagates the electromagnetic fields in the FDTD grid directly in the time domain by a small amount for every every time step.

Because it does this in an explicit fashion the time step cannot be assigned arbitrarily to any value we wish. It needs to be controlled and has an upper bound limit which is:

$$\Delta t \leq \frac{1}{c\sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}}}$$

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Courant Stability Condition

The 2D condition is easily obtained from the 3D one ...

$$\Delta t \leq \frac{1}{c\sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}}}$$

↓

$$\Delta t \leq \frac{1}{c\sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}}}$$

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Courant Stability Condition

When all spatial steps are equal:

$$\Delta x = \Delta y = \Delta z = \Delta l$$

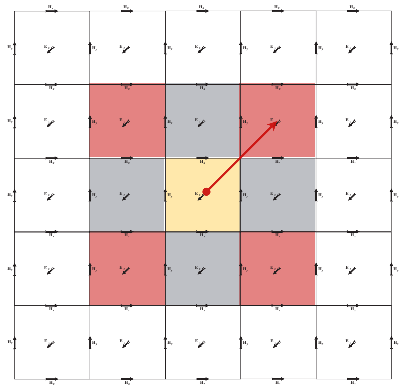
3D

$$\Delta t \leq \frac{\Delta l}{c\sqrt{3}}$$

2D

$$\Delta t \leq \frac{\Delta l}{c\sqrt{2}}$$

2D Stability: What does it mean?



$$t = 2\Delta t$$

$$L = \Delta l\sqrt{2}$$

$$c = \frac{L}{t} = \frac{\Delta L\sqrt{2}}{2\Delta t}$$

$$\Delta t \leq \frac{\Delta l}{c\sqrt{2}}$$

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So, the stability condition tells us that information in the model cannot travel faster than the speed of light! Something, that we did already know ...

The stability condition is formally obtained by substituting into the FDTD equations of a plane, monochromatic, traveling-wave trial solution.

This type of analysis leads us as well to the dispersion relation of the FDTD method and allow us to establish formally the stability criterion.

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Objects and media

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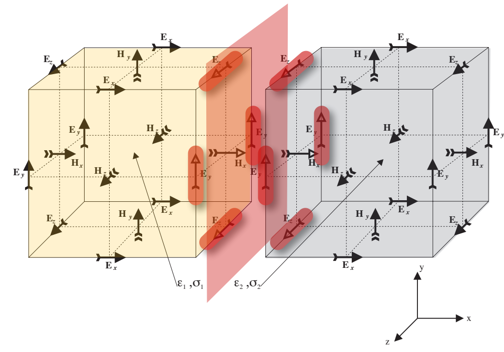
The FDTD lattice is a construction which represents locations of electromagnetic field components.

Building an object in the FDTD grid involves the assignment of constitutive properties at appropriate locations of electric and magnetic field components rather than filling Yee cells with material.

This sometimes creates a “conceptual” problem on how targets and objects are represented. It is not very significant for dielectric media but care is needed for perfect electric conductors.

3D Object modelling in FDTD

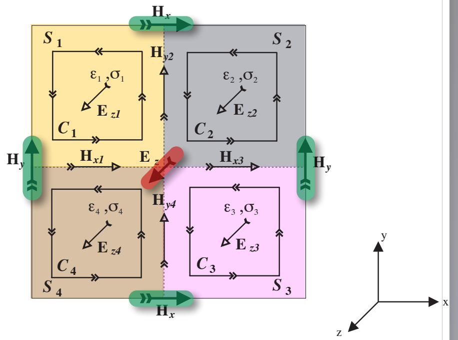


$$\oint_C \mathbf{H}\cdot d\hat{\mathbf{l}} = \iint_S \frac{\partial \mathbf{D}}{\partial t}\cdot d\hat{\mathbf{s}} + \iint_S \mathbf{J}_e \cdot d\hat{\mathbf{s}}$$

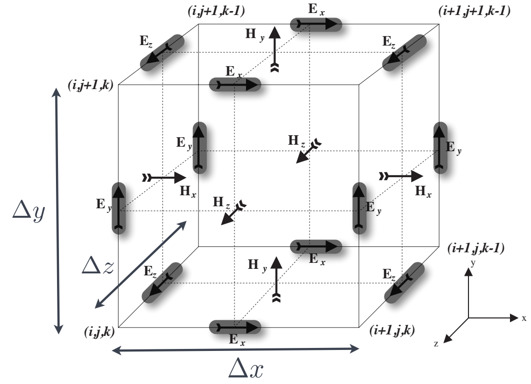
$$\sum_{i=1}^{4}\left(\oint_{C_i} H\cdot d\hat{l}\right) = \sum_{i=1}^{4}\left(\frac{\partial}{\partial t}\iint_{S_i}\epsilon_i E\cdot d\hat{S} + \iint_{S_i}\sigma_i E\cdot d\hat{S}\right)$$

$$\epsilon_{av} = \frac{\sum_{i=1}^{4}\epsilon_i S_i}{S}$$

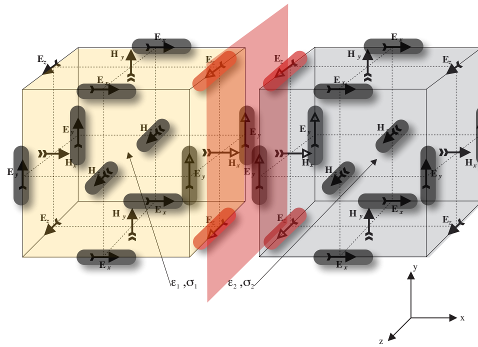
$$\sigma_{av} = \frac{\sum_{i=1}^{4}\sigma_i S_i}{S}$$



A cube of Perfect Electric Conductor



2D Object modelling in gprMax



If both magnetic and electric properties are varying then objects boundaries are not clearly defined as the electric and magnetic fields are staggered in space. Fortunately, this is not common practice in GPR.

The generalized Yee or "averaging" approach works well with dielectric materials. In benchmark problems (e.g. scattering of spheres) it produces more accurate results in comparison with the non-averaging scheme.

Boundary Conditions

- Electromagnetic Boundary Conditions
- Initial Conditions
- Problem specific Boundary Conditions

Electromagnetic Boundary Conditions

On the boundary between two media with finite but different electrical properties:

- The tangential electric fields are continuous.
- The tangential magnetic fields are continuous unless there is a surface current on the boundary which in this case are discontinuous by the amount: $-\hat{n} \times J_s$
- The electric flux density perpendicular to the interface is continuous unless there is a surface charge on the boundary which in this case is discontinuous by that amount: ρ_s
- The magnetic flux density perpendicular to the interface is continuous

Electromagnetic Boundary Conditions

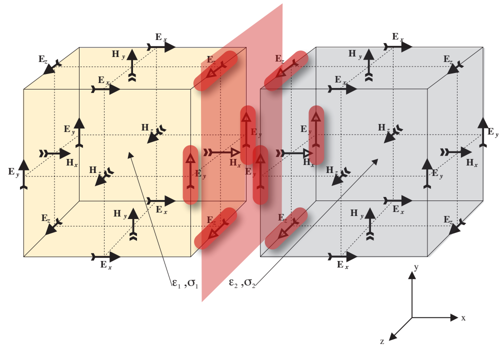
On the boundary between two media one of which is a perfect conductor:

- The tangential electric fields are zero.
- The tangential magnetic fields are equal to a surface current on the boundary which is: $-\hat{n} \times J_s$
- The magnetic flux density perpendicular to the interface is zero
- The electric flux density perpendicular to the interface equal

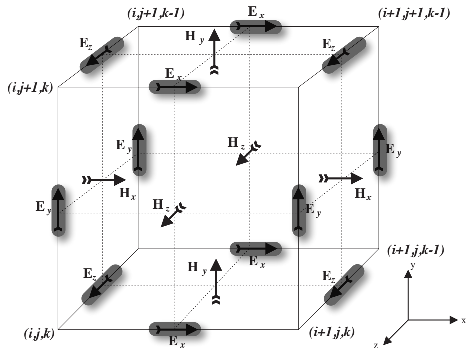
ρ_s

FDTD satisfies intrinsically the electromagnetic boundary conditions. In other words we don't need to do anything about them!

Boundary conditions in FDTD



BCs for Perfect Electric Conductor



Initial Conditions

The variation in space and time of our sources need to be specified as they are the initial conditions.

This is easily achieved directly in the time domain by specify source current densities at specific points in our model according to the Maxwell-Ampere equation:

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J_c} + \mathbf{J_s}$$

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$$\epsilon \frac{\partial E_y}{\partial t} + \sigma E_y + J_y = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}$$

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$$J_y = \frac{I_s(t)\Delta y}{\Delta x \Delta z}$$

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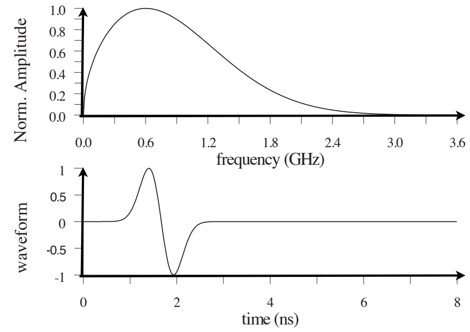
$$
\begin{aligned}
E_y\Big|_{(i,j+1/2,k)}^{n} &= \left(\frac{1-\frac{\sigma(i,j+1/2,k)\Delta t}{2\epsilon(i,j+1/2,k)}}{1+\frac{\sigma(i,j+1/2,k)\Delta t}{2\epsilon(i,j+1/2,k)}}\right) E_y\Big|_{(i,j+1/2,k)}^{n-1} + \\
&\left(\frac{\frac{\Delta t}{\epsilon(i,j+1/2,k)}}{1+\frac{\sigma(i,j+1/2,k)\Delta t}{2\epsilon(i,j+1/2,k)}}\right)\left(\frac{H_x\Big|_{(i,j+1/2,k+1/2)}^{n-1/2} - H_x\Big|_{(i,j+1/2,k-1/2)}^{n-1/2}}{\Delta z}\right. \\
&\left. - \frac{H_z\Big|_{(i+1/2,j+1/2,k)}^{n-1/2} - H_z\Big|_{(i-1/2,j+1/2,k)}^{n-1/2}}{\Delta x}\right) \\
&- \left(\frac{\frac{\Delta t}{\epsilon(i,j+1/2,k)}}{1+\frac{\sigma(i,j+1/2,k)\Delta t}{2\epsilon(i,j+1/2,k)}}\right)\left(\frac{\Delta y}{\Delta x \Delta y}\right) I_s^{n-1/2}
\end{aligned}
$$

Example of $I_s(t)$

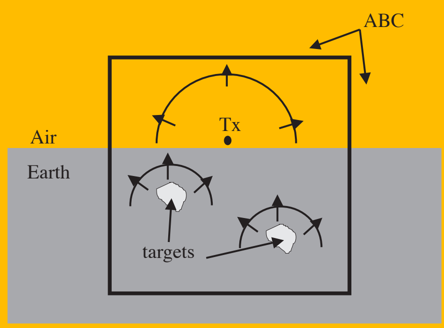


Problem specific Boundary Conditions

In most cases there are no specific boundary conditions that bound or limit the spatial extent of a GPR forward problem and can be used to prescribe specific values for the electromagnetic fields.

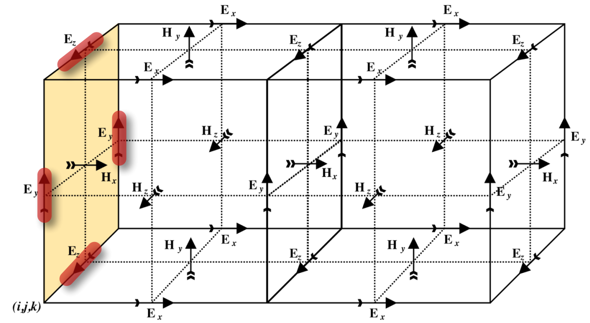
The boundary condition for the GPR problem is the **radiation condition** which means that the fields are decayed to zero values at infinity. This is tricky to accomplish with finite computational resources

Obviously, the FDTD grid which represents our model needs to be finite. This leads to the requirement for ....



Absorbing Boundary Conditions

We need ABCs ...



to compute the boundary field components!

Local ABCs (of historical interest)

Local absorbing boundary conditions: These are based on the "one wave equation". This is a pseudo-differential equation which aims to propagate waves in one direction only! In essence these conditions predict the required boundary field values from known internal values at previous and current time instant.

Their performance depends strongly on angle of incidence, the assumed velocity of propagation, and cannot be placed close to sources and scatterers. They have problems for layered geometries or strongly inhomogeneous models.

They exhibit instabilities when the order of approximation is increased. They are not used often any more!

Perfectly Matched Layers

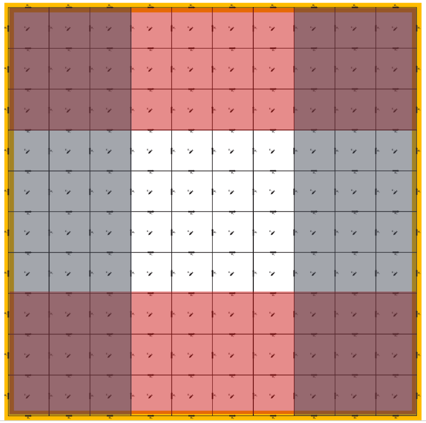
Perfectly Matched Layers: These are based on the introduction of special properties on the boundary layers of the FDTD grid. These layers are then "matched" to the normal FDTD grid properties. At the same time they attenuate the electromagnetic fields so no reflection returns from the FDTD grid boundary.

PMLs can be seen as non-Maxwellian anisotropic materials or as a scaling of space using a complex variable or otherwise as "complex stretching of space"

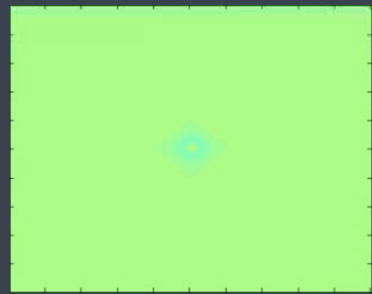
PMLs offer excellent absorption in most cases and their performance could be independent of frequency or underlying medium properties.

Local ABCs

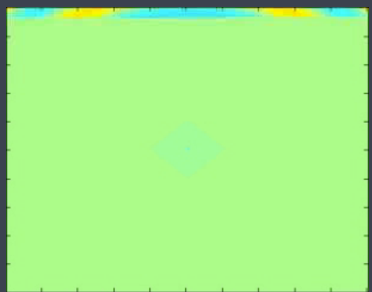
Perfectly Matched Layer



Simple harmonic source



Simple pulse source



Standard PML

Revolutionised ABCs in computational electrodynamics and other areas of wave propagation!

Relative simple to implement.

Does not work very well for evanescent and other inhomogeneous waves.

Does not work very well for grazing incidence waves and very low frequency modes.

$$\kappa + \frac{\sigma_{pml}}{j\omega}$$

CFS PML

Introduced to remove the zero frequency pole of the standard formulation.

More costly to implement especially for anisotropic medium PMLs

Works with evanescent waves and standard propagating modes.

Difficult to optimise! To many degrees of freedom.

$$\kappa + \frac{\sigma_{pml}}{\alpha + j\omega}$$

CFS PML formulations

Split field PML and complex stretched formulations are theoretically equivalent.

Implementation of non-split PMLs is much preferred for simplicity.

CFS PML implementations

The “Convolutional” PML (CPML) (Roden and Gedney)

The “Recursive Integration” PML (RIPML) (Giannopoulos)

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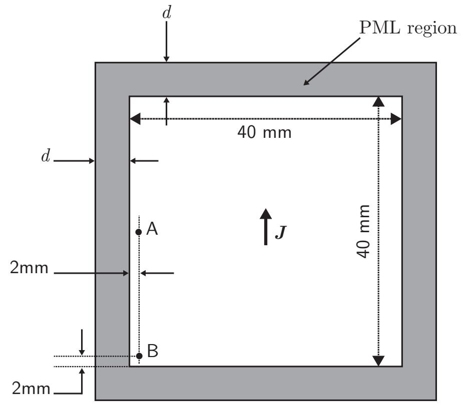

2D test

40 x 40 cells

$$I(t) = -2\frac{(t-t_0)}{t_w}e^{-(\frac{t-t_0}{t_w})^2}$$

$$t_w = 26.53\,\mathrm{ps}$$

$$t_0 = 4t_w$$

$$\mathrm{Error_{db}}|^n_{i,j} = 20\log_{10}\frac{\|E|^n_{i,j} - E_{\mathrm{ref}}|^n_{i,j}\|}{\|E_{\mathrm{ref_{max}}}|_{i,j}\|}$$

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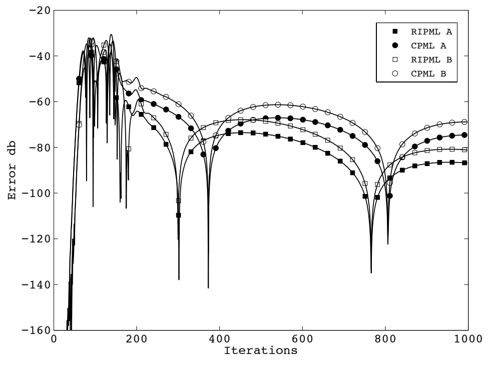

2D test

$$\kappa = 1$$
$$\alpha = 0$$
$$\sigma = \sigma_{opt}$$

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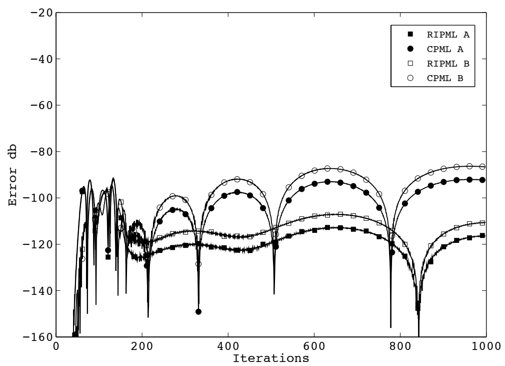
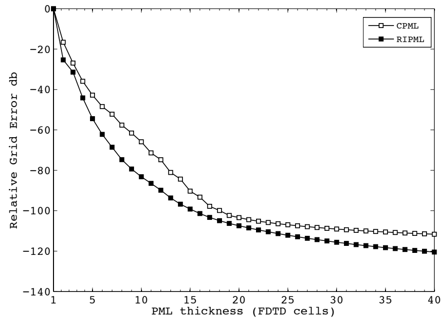
2D test



$$\kappa = 1$$
$$\alpha = 0.2$$
$$\sigma = \sigma_{opt}$$

2D test



$$\text{Grid Error}_{\text{db}} = \sum_{n=0}^{N}\sum_{i=0}^{NY}\sum_{j=0}^{NX} \frac{\| E|_{i,j}^{n} - E_{\text{ref}}|_{i,j}^{n} \|}{\| E_{\text{ref}_{\text{max}}}|_{i,j} \|}$$

Errors

... there is no free dinner either ...

In any numerical approximation there are errors which we are trying to minimize. Some are more important than others but it is paramount to always remember that a model is just an approximation!

We can make this approximation as good or bad as we wish depending on resources and time and obviously skill and understanding of the underlying issues that govern this approximation.

Truncation error

This type of error is only relevant if frequency domain quantities need to be calculated from the time domain output of the model.

It relates to the fact that the time domain response is of finite duration and if there is significant oscillations that have not reduced in amplitude by the time we stop the simulation the Fourier transformation of the time domain response will be affected.

This error is not of great significance for most GPR models!

Coarseness error

This relates to the choice of the spatial discretization step. It is a modelling error that creeps in if we do not resolve the geometry of a target adequately using enough Yee cells.

In essence this error means we are modelling the wrong target if we have not used enough Yee cells to resolve its important features properly.

Minimizing this error has an effect on computational requirements.

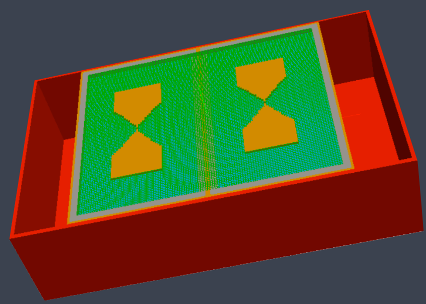
For example, modelling a rebar with 1 cm diameter using 1 cm spatial step ...

Staircasing error

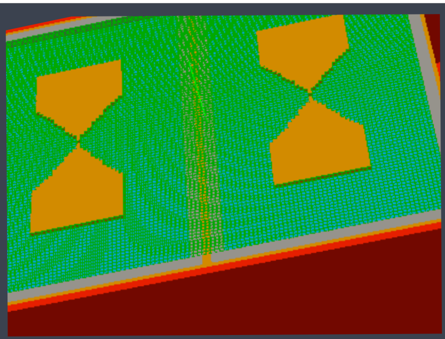
An FDTD grid is a collection of Yee cells. Objects that in have curved boundaries are approximated by a staircased version of their real geometry.

If the spatial step is small enough in comparison with the curvature of the object's boundary then this error should be small.

It is worst for curved electric conductors as these support currents on their surface. The path of these currents is not exactly the same in the model as it would be on the real target.



GSSI 1.5 GHz



GSSI 1.5 GHz

Velocity error (Numerical Dispersion)

This is the most significant of errors as it can introduce artificial dispersion of the electromagnetic pulses propagating in the FDTD grid.

The numerical phase velocity of propagation in the FDTD model deviates from the one in the actual medium by an amount which depends on the ratio of the wavelength to the spatial step. The greater this ratio the smaller the error.

This error depends on the direction of wave propagation as well!

after some maths ....

$$\left[\frac{1}{c\Delta t}\sin(\frac{\omega\Delta t}{2})\right]^2 = \left[\frac{1}{\Delta x}\sin(\frac{k_x\Delta x}{2})\right]^2 + \left[\frac{1}{\Delta y}\sin(\frac{k_y\Delta y}{2})\right]^2 + \xcancel{\left[\frac{1}{\Delta z}\sin(\frac{k_z\Delta z}{2})\right]^2}$$

which for small arguments of the sines

$$\left(\frac{\omega}{c}\right)^2 = k_x^2 + k_y^2 + \xcancel{k_z^2}$$

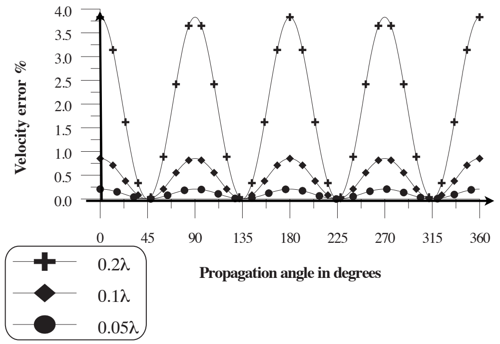
However, when ...

$$k_x = k_y = k_z = k/\sqrt(3)$$

and

$$\Delta t \leq \frac{\Delta l}{c\sqrt{3}}$$

There is no dispersion !!!



so, to keep this error at bay ...

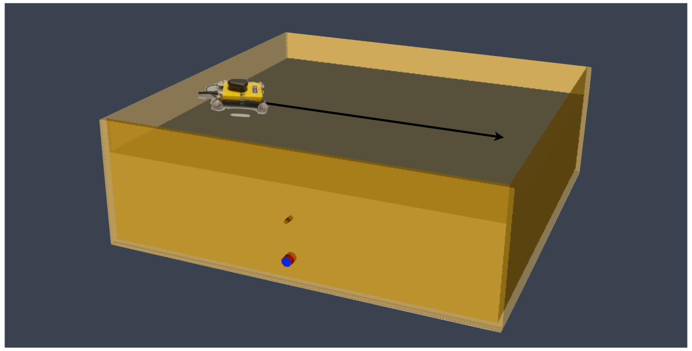
$$\Delta l \leq \frac{\lambda_{min}}{10}$$

at least ....

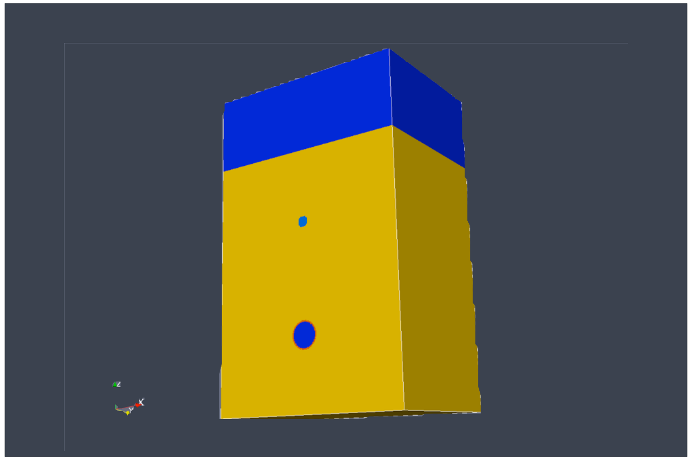
Advanced topics

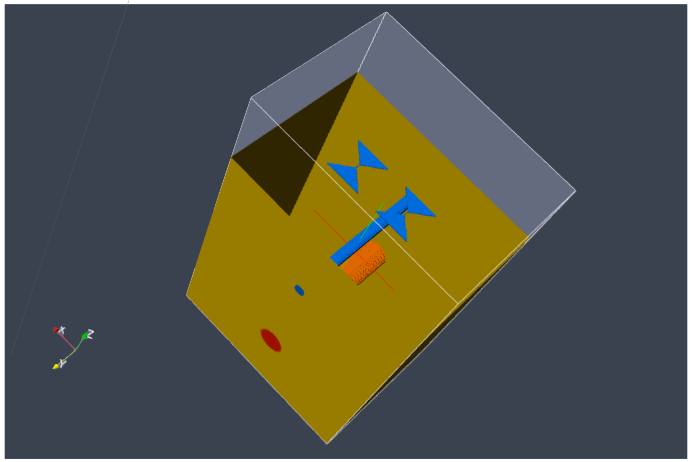
- Dispersive materials.
- Higher Order approximations to reduce the velocity error.
- Introducing sub-grids with finer spatial resolution only where is needed.
- Advanced Parallelisation using GPUs to increase performance. (something is smoking here . . .)

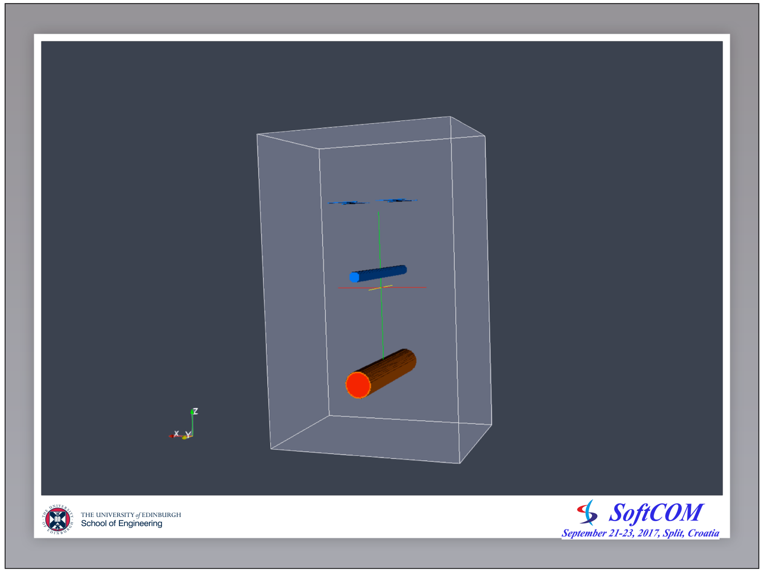
so, how we use all these stuff?



Simple Sandbox experiment



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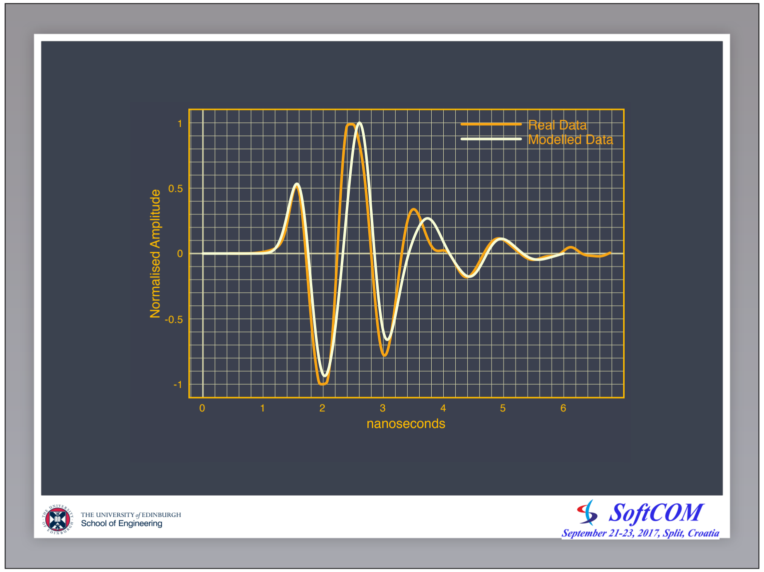
Real Data
Modelled Data
Normalised Amplitude
nanoseconds
1
0.5
0
-0.5
-1
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1
2
3
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5
6
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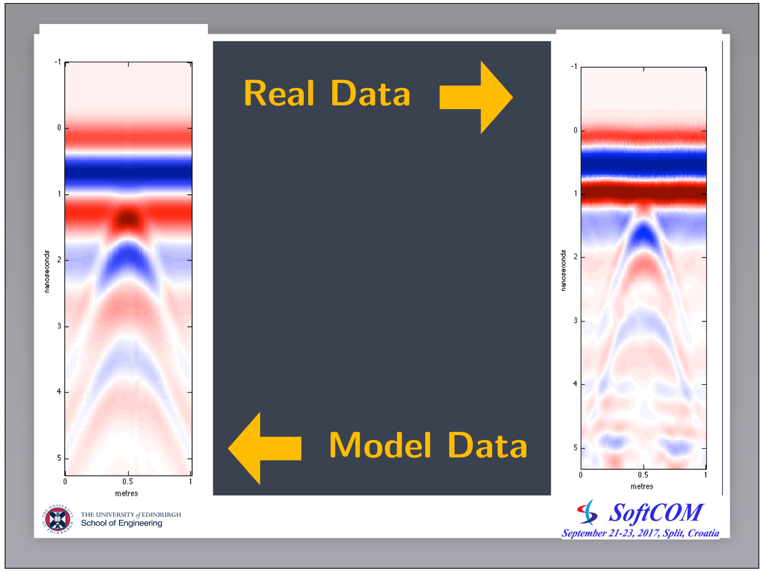
Real Data
Model Data
nanoseconds
metres
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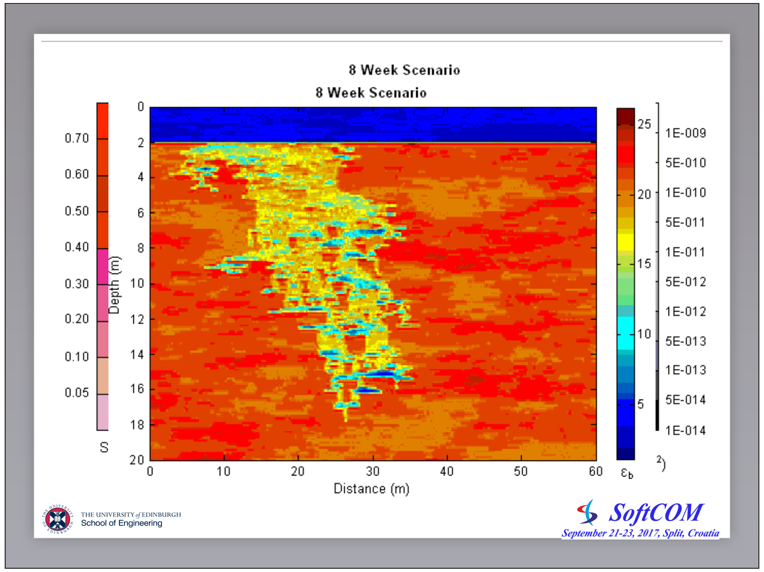
8 Week Scenario
8 Week Scenario
Depth (m)
Distance (m)
S
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.05
1E-009
5E-010
1E-010
5E-011
1E-011
5E-012
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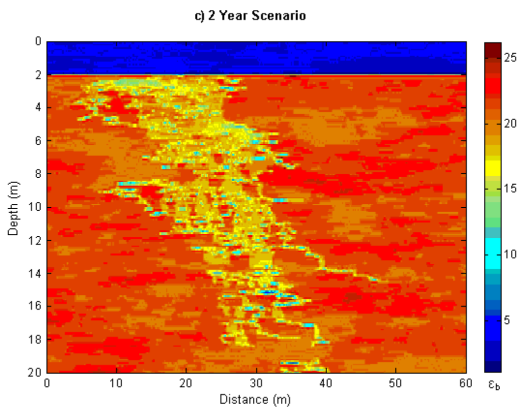

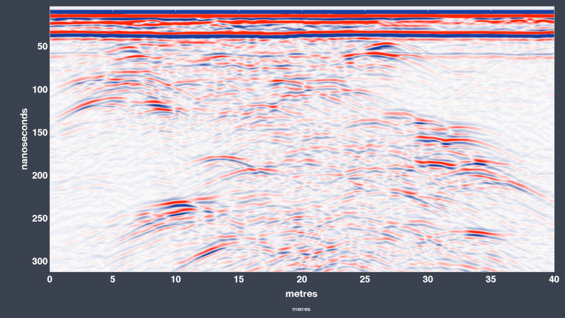

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... thats enough with the theory!

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... on to gprMax

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gprMax is a numerical model for GPR based on the FDTD method. It was made up by two different programs GprMax2D and GprMax3D for 2D (TMz) and 3D numerical modelling. Now the new version utilises only one code for both.

It all started in 1996 with support by the Building Research Establishment (BRE) as they wanted a GPR numerical model to simulate the responses from rebars and other defects in concrete. Thanks BRE!

Since then it was very slowly been upgraded and “enhanced” with some new features mainly by myself.

The old GprMax was written in C and ran from laptops to HPC platforms.

No need to reinvent the wheel...

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gprMax

the

good

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gprMax

the

bad

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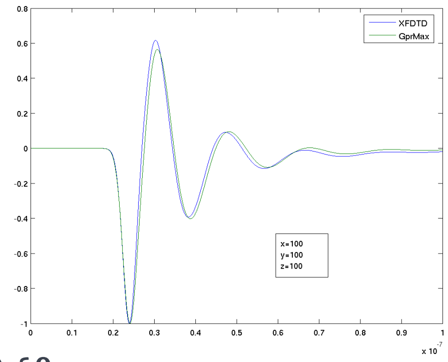
gprMax

and

the ugly

XFDTD @ $15,000



GprMax3D @ £0

The basics!

In gprMax all things happen with the aid of simple commands that are used to define the model parameters. These commands need to be entered in a simple ASCII (text!) file. You can use any of your favourite editors to do so.

Units and co-ordinates

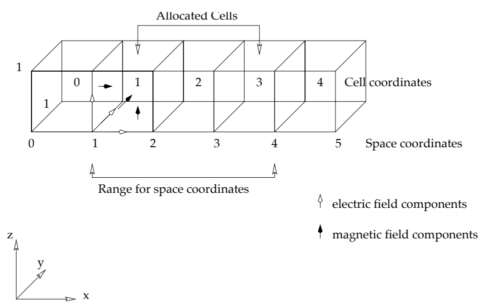
The SI unit system is used.

Although internally gprMax works on a FDTD grid made up from Yee cells objects are specified in real co-ordinates and not in cell numbers.

This simplifies rescaling of a model by just changing the spatial steps without having to recompute cell co-ordinates.

However, it is very important to remember that field components are not co-located - as explained before - and rounding is internally applied to the nearest cell.

gprMax



The time step is calculated according to the Courant stability condition and there is no need to worry about it.

Sources can be turned on and off and you can have as many as you like. GPR arrays?

Output points can be located anywhere in the model and don't have to be realistic! In other words you can have an output point inside an object to monitor the evolution of the fields.

Unless you model the antennas the sources mimic theoretical Hertzian dipoles and the receivers just appropriate field components.

Object creation in the model is done with the concept of a "painter's canvas". Objects that are specified later override the properties of the ones that were occupying the same cells previously.

The model space is always initialised to the properties of free space.

So, the order of object creation commands is important and you need to think about them at least for a little while ...

The order of other commands is not important but there are other exceptions as new features are added!

In gprMax all commands start with a # as the first character in the line containing the command.

Only one command per line is allowed.

Some common pitfalls ...

Not really modelling what you really think you are modelling!

Specifying sources and output points (receivers) in or extremely close to the PML!

Too coarse models for the targets you want to model.

Too coarse models for the size of the wavelength of the maximum frequency. This leads to numerical dispersion.

Thinking that field components are specified at the same point and surprised by the output!

Not including enough free space cells above the air-earth interface.

. . .not reading the documentation!

and remember ...

Garbage in, Garbage out!

Future features!

- Subgrids for finer resolution only where needed.
- 4th order boxes to control numerical dispersion in high value dielectrics.
- More object generation primitives.
- Benchmarking. Stop and restart simulations.
- Plane wave excitation that takes into account a half space or layered earth model.
- Any reasonable requests - we accept funding and large donations ;-)

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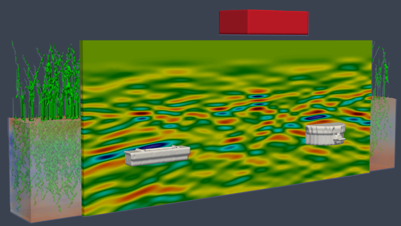
so, if you do it well you can do . . .



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Thanks for licensing!

Questions?

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